Eton College King's Scholarship Examination 2010

(One and a half hours)

MATHEMATICS A

Answer Question 1 and as many of the other five questions as you can.
Question 1 is worth 50 marks. All other questions are worth 10 marks each.

Show all of your working. The use of calculators is permitted.

1. **Compulsory Question**

(a) If $a=5$, $b=2$ and $c=-4$ then evaluate the following:

(i) $5a+3b+c$ [1]

(ii) $a(b-c)$ [1]

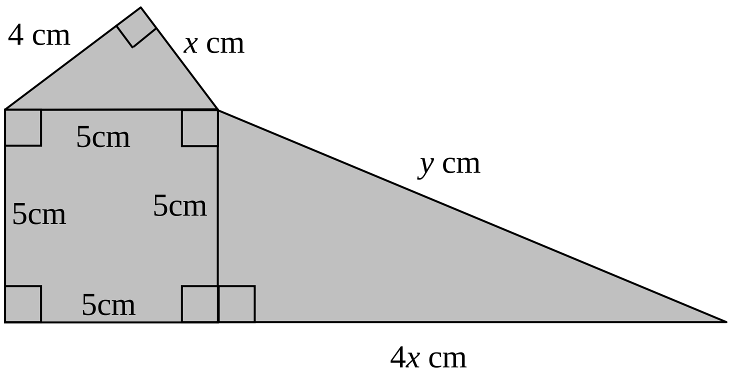
(iii) a^2-c^2 [1]

(b) Solve the following inequalities:

(i) $5x-7<11$ [1]

(ii) $\dfrac{3-2x}{7}\geq 4$ [2]

(c) The diagram shown below consists of a square of length 5cm and two right-angled triangles. Both triangles have one side of length 5cm.



(i) By considering the smaller triangle, show that $x=3$. [1]

(ii) Hence find the value of y. [2]

(iii) Calculate the external perimeter of the shape. [2]

(iv) Calculate the area of the shape. [2]

(d) The three angles in a triangle are $5n^\circ$, $2n^\circ$ and $(n+20)^\circ$.

(i) Find the value of n. [2]

(ii) What sort of triangle is this? [1]

(e) Expand and simplify the following:

(i) $(x-2)(x+2)$ [1]

(ii) $(x-3)(x^2+3x+9)$ [2]

(f) Solve the following equations:

(i) $\frac{3x-1}{2}=13$ [2]

(ii) $\frac{3x-1}{2}-\frac{x-3}{4}=9$ [3]

(g) Showing all working clearly, simplify the following as far as possible:

(i) $7\frac{2}{3}+2\frac{1}{5}$ [2]

(ii) $\frac{1}{x}+\frac{1}{2x}$ [2]

(h) Showing all working clearly, express the following as single fractions:

(i) $\frac{2\frac{1}{3}}{3\frac{1}{2}}$ [2]

(ii) $\frac{a+\frac{1}{b}}{b+\frac{1}{a}}$ [3]

(i) Given the formula $s=\frac{1}{2}(u+v)t$:

(i) find v when $s=50$, $t=4$ and $u=10$, [1]

(ii) find u in terms of s, v and t. [2]

(j) In a news report it was stated that the price of a train ticket from Manchester to London had risen to £264.50. The newsreader then said "this is up 15 from last year". She did not state whether she meant "up 15%" or "up £15". Find the difference between the two alternatives. [3]

(k) Simplify the following:

(i) $5x^3+15x^2$ [1]

(ii) $\frac{16x^3+4x^2}{8x^2+2x}$ [2]

(l) Solve the following simultaneous equations:

$3x+7y=13$

$5x-3y=29$ [4]

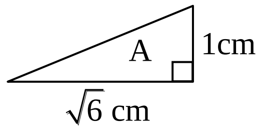
(m) Five numbers a, b, 15, 18, c are listed in increasing order (though two or more numbers next to each other may be equal).

The mode of these five numbers is 15, the mean is 16 and the range is 10.

Find the values of a, b and c.

[4]

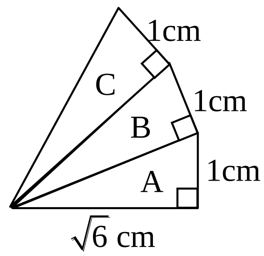
2. A right-angled triangle A with a base of length $\sqrt{6}$ cm and a height of 1cm is drawn so that its base is horizontal.



(a) Find, in the form $\sqrt{k}$ (where k is an integer), the hypotenuse of triangle A. [2]

A right-angled triangle B, also of height 1cm, is placed on top of A so that its base is the hypotenuse of A.

A right-angled triangle C, also of height 1cm, is placed on top of B so that its base is the hypotenuse of B.



(b) Find the exact length of the hypotenuse of triangle C. [3]

The same procedure as above is repeated but instead of the initial triangle having a base of length $\sqrt{6}$ cm it has a base of length 45cm and instead of the triangles all having heights of 1cm, they all have heights of $\sqrt{k}$ cm where k is a positive integer.

(c) After a series of more than one, but fewer than ten, of these triangles the hypotenuse of the last triangle is 52cm. Find how many triangles there are in this series, including the initial triangle and find also the value of k. [5]

3. $a \times a \times a \times a$ can be expressed more simply as a^4.

(a) Explain briefly why $\left(3^4\right)^3 = 3^{12}$. [1]

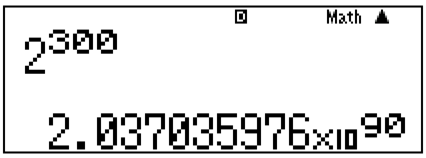
(b) (i) Write down the expression $3^4.....10^2$ replacing the "…" with either a "<" or a ">" sign. [2]

(iii) Explain, without using the calculator, why 3^{12} is less than one million. [2]

(c) (i) Calculate 2^{10}, 5^6 and 10^3. [1]

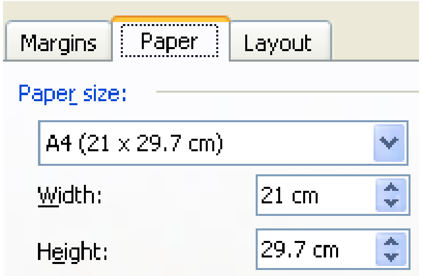
(ii) Put the three numbers 2^{400}, 5^{180} and 10^{120} in ascending order, giving reasons for your answer. [2]

(d) Lydia wants to know how many digits there are in 2^{1200} when it is written out as an integer. The calculator gives her an error message for any power of 2 beyond 2^{332}. The calculator gives her the following result.



Use the above calculator screenshot to calculate how many digits there are in 2^{1200} when it is written out as an integer. [2]

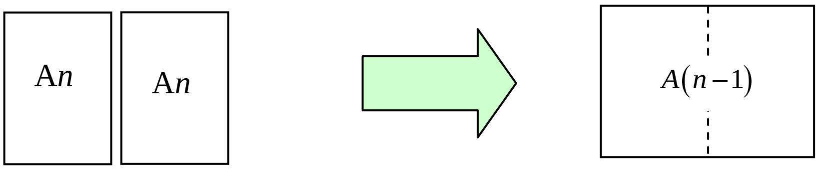
4. The following computer screenshot from Microsoft Word states the approximate dimensions of an A4 piece of paper:



Two of the properties of the A series of paper are as follows:

Property 1 The A0 piece of paper has an area of $1\ \text{m}^2$.

Property 2 For positive integers *n* when two pieces of A*n* paper are placed together with their longer sides touching, an $A(n-1)$ piece of paper is formed, as shown below.

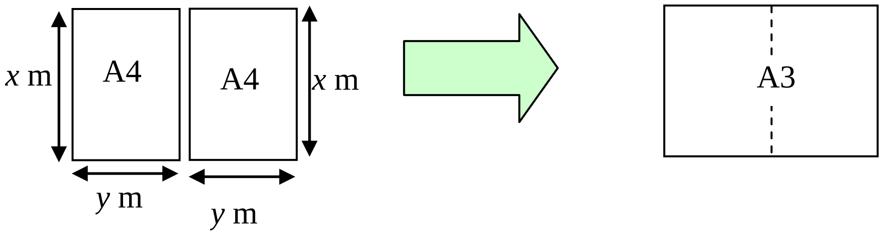


(a) Use these properties to:

(i) find, as a fraction, the area (in m^2) of an A1 piece of paper, [1]

(ii) show that the area of an A3 piece of paper is $\frac{1}{8}\text{m}^2$. [1]

Suppose that the dimensions of an A4 piece of paper are *x* m by *y* m. When two pieces of A4 paper are placed together with their longer sides touching, an A3 piece of paper is formed, as shown below.



(b) Write down, in terms of *x* and *y*, the width and length of the A3 piece of paper. [1]

(c) Use part (b) and (a)(ii) to find *y* in terms of *x*. [2]

The third property of the A series of paper is as follows:

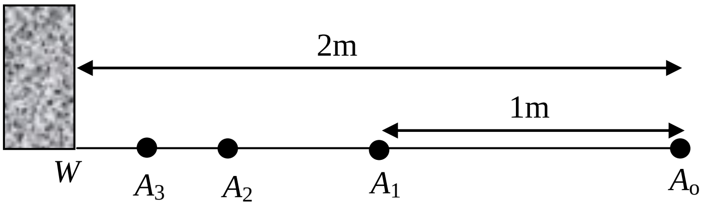
Property 3 The ratio of the width to length in an A*n* piece of paper is the same for every value of *n*.

(d) Use property 3 to find the integer value of *m* such that $y^2 = \frac{x^2}{m}$. [1]

(e) Solve the equations in (c) and (d) to find *x* (to 4dp). [2]

(f) Hence find the dimensions (in cm) of an A4 piece of paper (to 2dp). [2]

5. (a) Joshua stands on a horizontal floor at the point A_o which is 2m away from the foot of a wall, W. He walks half the distance to the wall and then stops at the point A_1. He repeats this process a further two times until he reaches the point A_3.



(i) How far is Joshua from the wall when he is at A_3? [1]

$x = 1 + \frac{1}{2} + \frac{1}{4}$ represents the total distance he has walked.

(ii) By considering where he is in relation to the wall, express x as a fraction. [2]

If Joshua were able to continue this process for an infinite number of times then the distance he would have walked could be expressed as $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$.

(iii) By considering where he would be in relation to the wall, write down the value of S. [2]

(b) Joshua now considers another infinite sum T where

$T = \frac{1}{2} + \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{5}{32} + \frac{8}{64} + \ldots\ldots$

The numerator of each subsequent term is the sum of the numerators of the previous two terms. The denominator of each subsequent term is twice the denominator of the previous term.

To make this pattern obvious, Joshua does not cancel down any of the fractions in this sum.

(i) By considering the first six terms of $\frac{1}{2}T$ and $\frac{1}{4}T$, find the values of a, b, c, d, e and f where $\frac{3}{4}T = \frac{a}{4} + \frac{b}{8} + \frac{c}{16} + \frac{d}{32} + \frac{e}{64} + \frac{f}{128} + \ldots$. [2]

(ii) Hence find the value of T. [3]

6. A craftsman has two identical cylinders of wood, both of radius 6cm and height 6cm. In each cylinder the point *O* is in the middle of the circular top surface.

With the first cylinder the craftsman hollows out a hemisphere (half a sphere) of radius 6cm, as shown in Fig 1, so that a bowl shape remains.

The point X is vertically below O and the point A lies on the curved surface of the wood on the same horizontal level as X.

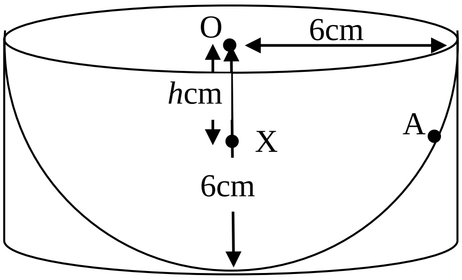


Fig 1

The craftsman then cuts horizontally through the bowl shape, through the point X. He discards the top piece and keeps only the bottom piece.

The shape of the horizontal surface of the bottom piece (that is the wood on the same horizontal level as X) is shaded in Fig 2.

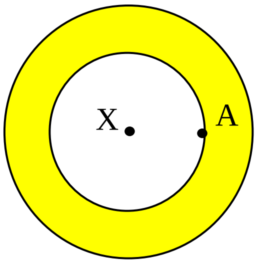


Fig 2

(a) Write down the length of OA. [1]

(b) Calculate, in terms of *h*, the length *XA*. [2]

(c) Find the area of this shaded surface, in the general case, giving your answer in terms of *h* and π. [3]

With the second cylinder the craftsman cuts away wood so that the shape that remains is a cone of radius 6cm and height 6cm, as shown in Fig 3.
The point X is vertically below O and the point A lies on the curved surface of the wood on the same horizontal level as X.

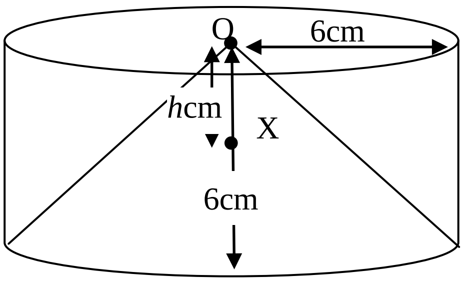


Fig 3

As before, the craftsman then cuts horizontally through the cone, through the point X. As before, he discards the top piece and keeps only the bottom piece.

(d) Find the cross-sectional area of the horizontal surface of the bottom piece (that is the wood on the same horizontal level as X), giving your answer in terms of *h* and π. [2]

Alice thinks that the volumes of the two bottom pieces will be equal for all values of *h*.
Florence thinks that the volumes of the two bottom pieces will be equal for only certain values of *h*.

(e) State, with a brief reason, whether Alice or Florence is correct. [2]

END OF PAPER